The harmonic map and Killing fields for self-dual SU(3) Yang-Mills equations

1984 J. Phys. A: Math. Gen. 17 L227

(http://iopscience.iop.org/0305-4470/17/5/001)

J. Phys. A: Math. Gen. **17** (1984) L227–L230. Printed in Great Britain

LETTER TO THE EDITOR

The harmonic map and Killing fields for self-dual SU(3) Yang–Mills equations

Paul Kersten and Ruud Martini

Department of Applied Mathematics, Twente University of Technology, PO Box 217, 7500 AE Enschede, the Netherlands

Received 9 December 1983

Abstract. Using symbolic computations, the unique metric in the space of fields, required to describe self-dual SU(3) Yang–Mills equations by a harmonic map, is determined. Moreover the complete Lie algebra of Killing fields for this metric is established.

In the search for complete integrability of partial differential equations arising in mathematical physics, Xanthopoulos (1981, 1982) discussed the complete integrability of self-dual SU(3) Yang–Mills equations by using harmonic maps between Riemannian spaces.

Xanthopoulos (1981) demonstrated that the metric g'_{AB} of the Riemannian space (N, g'_{AB}) with line element in the local chart $\{f^A\} = (u, v, x_1, x_2, y_1, y_2, w_1, w_2)$ given by

$$\begin{aligned} ds^2 = g'_{AB}(df^A)(df^B) &= u^{-2}(du)^2 + v^{-2}(dv)^2 - (uv)^{-1}(du)(dv) + uv^{-2}(dy_1)(dy_2) \\ &\quad + (y_1 y_2 (uv)^{-1} + vu^{-2})(dx_1)(dx_2) - y_1(uv)^{-1}(dx_1)(dw_2) \\ &\quad - y_2(uv)^{-1}(dx_2)(dw_1) + (uv)^{-1}(dw_1)(dw_2) \end{aligned} \tag{1}$$

is strongly related to self-dual SU(3) Yang–Mills equations. In fact, a reduced system of Yang–Mills equations is just the harmonicity condition with respect to the metric g'_{AB}. Xanthopoulos (1982) constructs 16 linearly independent Killing fields for the metric g'_{AB} (1), accompanied by the following comment:

'We should mention at this point that we do not have a proof that we have found all the Killing fields of the metric g'_{AB}'.

We shall first show how the metric g'_{AB} can be determined in a *unique* way from the Christoffel symbols appearing as coefficients in the R-gauge form of self-dual SU(3) Yang–Mills equations. We shall then construct the overdetermined system of partial differential equations which has to be satisfied by the Killing fields i.e., the Killing equations. We determined the general solution of the Killing fields, thus proving the result, that the complete Lie algebra of Killing fields for the metric g'_{AB} is 16-dimensional, generated by the 16 independent fields constructed by Xanthopoulos (1982). The construction of the unique metric g'_{AB} and the determination of the general solution of the Killing fields have been achieved by symbolic computations in a semi-automatic way using software developed in the symbolic language Reduce and which has been described in Gragert *et al* (1981, 1983) and Kersten and Gragert (1983).

The determination of the metric g'_{AB} from the coefficients in the R-gauge representation follows. The starting point is the R-gauge representation of self-dual SU(3)

0305-4470/84/050227+04\$02.25

Yang-Mills equations given by Prasad (1978)

$$D_a\bar{D}_a u = u^{-1}(D_a u)(\bar{D}_a u) - vu^{-1}(D_a x_1)(\bar{D}_a x_2) - v^{-1}(D_a w_1 - y_1 D_a x_1)(\bar{D}_a w_2 - y_2 \bar{D}_a x_2),$$
$$D_a\bar{D}_a v = v^{-1}(D_a v)(\bar{D}_a v) - uv^{-1}(D_a y_1)(D_a y_2) - u^{-1}(D_a w_1 - y_1 D_a x_1)(\bar{D}_a w_2 - y_2 \bar{D}_a x_2),$$
$$D_a\bar{D}_a x_1 = (D_a x_1)[\bar{D}_a \ln(u^2/v)] + uv^{-2}(D_a w_1 - y_1 D_a x_1)(\bar{D}_a y_2),$$
$$D_a\bar{D}_a x_2 = (\bar{D}_a x_2)[D_a \ln(u^2/v)] + uv^{-2}(\bar{D}_a w_2 - y_2 \bar{D}_a x_2)(D_a y_1),$$
$$D_a\bar{D}_a y_1 = (D_a y_1)[\bar{D}_a \ln(v^2/u)] - vu^{-2}(D_a w_1 - y_1 D_a x_1)(\bar{D}_a x_2),$$
$$D_a\bar{D}_a y_2 = (\bar{D}_a y_2)[D_a \ln(v^2/u)] - vu^{-2}(\bar{D}_a w_2 - y_2 \bar{D}_a x_2)(D_a x_1),$$
$$D_a\bar{D}_a w_1 = (D_a x_1)(\bar{D}_a y_1) + uy_1 v^{-2}(D_a w_1 - y_1 D_a x_1)(\bar{D}_a y_2) + (D_a w_1)[\bar{D}_a \ln(uv)] + y_1(D_a x_1)[\bar{D}_a \ln(u/v^2)],$$
$$D_a\bar{D}_a w_2 = (\bar{D}_a x_2)(D_a y_2) + uy_2 v^{-2}(\bar{D}_a w_2 - y_2 \bar{D}_a x_2)(D_a y_1) + (\bar{D}_a w_2)[D_a \ln(uv)] + y_2(\bar{D}_a x_2)[D_a \ln(u/v^2)], \qquad (2a)$$

which can be denoted in a compact form as

$$D_a\bar{D}_a f^A + \Gamma^A_{BC}(D_a f^B)(\bar{D}_a f^C) = 0 \qquad (2b)$$

where $D_a = (\partial_y, \partial_z)$, $\bar{D}_a = (\partial_{\bar{y}}, \partial_{\bar{z}})$, and Γ^A_{BC} are easily obtained from (2*a*).

Let (M, g_{mn}) and (N, g'_{AB}) be two Riemannian spaces. In the theory of harmonic mappings, the mapping $f: M \to N$ is harmonic iff $f = \{f^A\}$ satisfies the condition

$$g^{mn}(D_m D_n f^A) + \Gamma^A_{BC}(D_m f^B)(D_n f^C) g^{mn} = 0 \qquad (3)$$

where $A, B, C = 1, \ldots, \dim N$; $m, n = 1, \ldots, \dim M$; Γ^A_{BC} are the Christoffel symbols for the metric g'_{AB}.

Remark. Note that by taking $y = \bar{y}$, $z = \bar{z}$, (2) reduces to (3).

Comparing system (2*b*) and (3) we now want to construct the metric g'_{AB} in such a way that Γ^A_{BC} are just the coefficients of the quadratic terms in (2); i.e.,

$$\Gamma^A_{BC} = \tfrac{1}{2} g'^{AS}(\partial_B g'_{CS} + \partial_C g'_{BS} - \partial_S g'_{BC}) \qquad (A, B, C = 1, \ldots, 8). \qquad (4)$$

Multiplication by g'^{RA} and summation yields

$$g'_{RA}\Gamma^A_{BC} = \tfrac{1}{2}(\partial_B g'_{CR} + \partial_C g'_{BR} - \partial_R g'_{BC}) \qquad (R = 1, \ldots, 8, B = 1, \ldots, 8, C = B, \ldots, 8). \qquad (5)$$

i.e., an *overdetermined system* of $8 \times \frac{1}{2} \times 8(8+1) = 288$ partial differential equations for the $\frac{1}{2} \times 8(8+1) = 36$ unknown functions g'_{AB} ($A = 1, \ldots, 8$, $B = A, \ldots, 8$; due to symmetry conditions).

We solved this overdetermined system in a rather 'straightforward' way, using the software described in Gragert *et al* (1981, 1983) and Kersten and Gragert (1983). For all details of the computation we refer to Kersten and Martini (1983).

We obtained the following result:

Theorem 1. The metric g'_{AB} (1) is determined in a *unique* way from the coefficients in the R-gauge representation (2).

We now construct the Killing equations for the metric g'_{AB} given in coordinates $(u, v, x_1, y_1, w_1, x_2, y_2, w_2)$ by the symmetric (8×8)-matrix

$$g'_{AB}\begin{bmatrix} \frac{1}{u^2} & -\frac{1}{2uv} & * & * & * & * & * & * \\ -\frac{1}{2uv} & \frac{1}{v^2} & * & * & * & * & * & * \\ * & * & * & * & * & \frac{1}{2}\left(\frac{y_1y_2}{uv}+\frac{v}{u^2}\right) & * & \frac{-y_1}{2uv} \\ * & * & * & * & * & * & \frac{u}{2v^2} & * \\ * & * & * & * & * & \frac{-y_2}{2uv} & * & \frac{1}{2uv} \\ * & * & \frac{1}{2}\left(\frac{y_1y_2}{uv}+\frac{v}{u^2}\right) & * & \frac{-y_2}{2uv} & * & * & * \\ * & * & * & \frac{u}{2v^2} & * & * & * & * \\ * & * & \frac{-y_1}{2uv} & * & \frac{1}{2uv} & * & * & * \end{bmatrix}. \tag{6}$$

We adopt the notation in Lovelock and Rund (1975).

The Killing equations of a metric in a n-dimensional space can be described by a set of $\frac{1}{2}n(n+1)$ partial differential equations for the functions $\xi_i (i=1,\ldots,n)$, being the covariant components of the Killing field $\xi^i\partial_{x_i}$ (summation convention $i=1,\ldots,n$), and which depend on the variables $x_1,\ldots,x_n$.

Let

$$ds^2 = g_{ij}\,dx^i\,dx^j, \tag{7}$$

then the Killing field $K=\xi^i\partial_{x_i}$ has to satisfy the following equations

$$\xi_{i,j}+\xi_{j,i}=0 \qquad (i=1,\ldots,n; j=1,\ldots,n) \tag{8}$$

where

$$\xi_k = g_{ik}\xi^i \qquad g_{ij}g^{jk}=\delta_i^k \tag{8a, b}$$

$$\xi_{k,j}=\partial_j\xi_k - \xi_1\Gamma^1_{kj} \qquad \text{(the covariant derivative)} \tag{8c}$$

$$\Gamma^i_{lk}=\tfrac{1}{2}g^{ij}(\partial_1 g_{kj}+\partial_k g_{lj}+\partial_j g_{kl}) \qquad \text{(Christoffel symbols).} \tag{8d}$$

In this case, where (g_{ij}) is given by (6), (8) yields an overdetermined system of $\frac{1}{2}\times 8(8+1)=36$ equations for the functions $\xi_i (i=1,\ldots,n)$. Again, this system of partial differential equations has been solved by the above mentioned software. For full details of the construction the reader is referred to Kersten and Martini (1983).

We obtain as a final result:

Theorem 2. The complete Lie algebra of Killing fields for the metric g'_{AB} (2.1) is 16 dimensional, where the generators are exactly those obtained by Xanthopoulos (1982).

In conclusion, using symbolic computations we have been able to determine the unique metric in the space of fields and to establish the complete Lie algebra of Killing fields for this metric.

References

Gragert P K H 1981 *PhD thesis*, Twente University of Technology

Gragert P K H, Kersten P H M and Martini R 1983 *Acta Applicandae Mathematicae* **1** 43–77

Kersten P H M and Gragert P K H 1983 *Symbolic integration of overdetermined systems of partial differential equations, Memorandum 430, Twente University of Technology*

Kersten P H M Martini R 1983 *The harmonic map and Killing fields for self-dual* SU(3) *Yang–Mills equations, Memorandum 443, Twente University of Technology*

Lovelock D and Rund H 1975 *Tensors, Differential Forms and Variational Principles Pure and Applied Mathematics* (New York: Wiley)

Prasad M K 1978 *Phys. Rev.* D **17** 3243–46

Xanthopoulos B C 1981 *J. Phys. A: Math. Gen.* **14** 1445–55

—— 1982 *J. Phys. A: Math. Gen.* **15** L61–6